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Transition to chaos in Bose-Einstein condensates: role of inter-component interactions on wavepacket breathing

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閉じ込めポテンシャルを持つ２次元の多成分 BEC のダイナミクスを、集団座標の方法の観点から考察する。一成分 BEC のダイナミクスではエーレンフェストの定理が成立し、ガウス型波束は安定であり波束幅は特徴的振動数で振動する。しかし、成分間に非線形相互作用を取り入れると、波束幅は不安定成長して波束間干渉が生じ、多成分 BEC に対する粒子描像が完全にこわれる。しかし、振動型の成分間非線形相互作用を採用すると、ガウス型波束は安定性を回復する。さらに、振動型相互作用の振幅を増加させると規則振動からカオス振動への遷移が生じる。3 成分 BEC のダイナミクス (3 体問題) を例にとり、このシナリオを具体的に示す。

Recently a great number of theoretical and experimental efforts have been devoted to Bose-Einstein condensates (BECs). As well as single-component BECs, the trapping techniques can create multi-component condensates which involve inter-component nonlinear interactions. The multi-component BEC, far from being a trivial extension of the single-component one, presents novel and fundamentally different scenarios for its ground state and excitations. In particular, it has been observed that BEC can reach an equilibrium state characterized by the separation of the species in different domains.

BEC has a dual aspect of waves and particles. The wave nature is high-lightened in the phenomenon of interference leading to fringe patterns. On the other hand, the particle nature of BECs can be seen in typical localized states like vortices and solitons. In fact solitons were observed in the quasi-one dimensional BEC[1, 2].

Among the works that emphasize a role of the particle picture for BEC in high-dimensions, those of Pérez-García's group are the most noteworthy[3–6]. We focus on two important assertions by their group. The first assertion made for a single-component BEC with a harmonic potential is as follows[5]: If the phase of BEC wave function will be suitably corrected, the center of mass for a wavepacket obeys Newtonian dynamics and Ehrenfest's theorem is valid even for the nonlinear Schrödinger equation (NSE). Furthermore the center of mass is decoupled from dynamics of the shape of a wavepacket. The second assertion is concerned with the multi-component BEC with a harmonic trap[6]: Under the condition that the distance between wavepackets associated with individual components is much larger than their typical widths, the particle picture still holds for the multi-component BEC and dynamics of interacting wavepackets is replaced by that for interacting centers of mass. The above condition is satisfied so long as the centrifugal force due to non-vanishing angular momentum competes well with the harmonic trap and guarantee the suitable separation between wavepackets.

Although the second assertion of Pérez-García's group is much more interesting than the first one, we have several criticisms: (1) Degrees of freedom for the width and phase of wave packets are not taken into consideration. As stated repeatedly by Pérez-García's group in other works of their own[3, 4], the width and phase are affected by the nonlinearity and vary as time elapses; (2) Nonlinear inter-component interaction (NICI) has a tendency to quickly broaden the width of individual wavepacket and breaks a many particles picture for the multi-component BEC, as will be evidenced below. However, by overcoming the above problems, we wish to propose a model for conservative chaos with a finite degrees-of-freedom emanating from the multi-component BEC in high dimensions.

In this talk, by applying the variational principle, we derive an effective nonlinear dynamics with a finite degrees of freedom from 2-*d* multi-component BEC with a harmonic trap. Then we show how the non-vanishing NICI makes the breathing of individual wavepackets unstable. In particular we shall investigate the transition of wavepacket breathing from regular to chaotic oscillation which is caused by increasing the amplitude of the time periodic NICI. We numerically analyze the subject on the basis of the "three-body problem" corresponding to three-component BEC.

The multi-component BEC at zero temperature is described by the nonlinear Schrödinger equation (NSE) (or Gross-Pitaevskii(GP) equation). We shall consider a system of n complex fields $\psi_1(t,\mathbf{r}), \psi_2(t,\mathbf{r}), \ldots, \psi_n(t,\mathbf{r})$ ruled by the equations

$$i\partial_t\psi_j(\mathbf{r}) = \left[-\frac{1}{2}\Delta_{\mathbf{r}} + V(\mathbf{r})\right]\psi_j(\mathbf{r}) + \sum_j U_j(t,\mathbf{r})\psi_j(\mathbf{r}) \tag{1}$$

for $j = 1, \ldots, n$ (in units of atomic mass $m = 1$ and confining length $\sqrt{\frac{\hbar}{m\omega}} = 1$). $V(\mathbf{r}) = \mathbf{r}^2/2 = (x^2+y^2)/2$ stands for a harmonic trap and $U_j(t,\mathbf{r}) = \sum_k g_{jk}|\psi_k(t,\mathbf{r})|^2$ is the nonlinear term. The coefficient is $g_{ij} = 4\pi\hbar^2 a$, where a is the scattering length tunable by Feshbach resonance. We consider the case of repulsive nonlinearity, $g_{ij} > 0$ and choose $g_{ii} = g$ and $g_{ij} = \Lambda g$.

In the absence of the inter-component interaction, each component has stationary states, namely Gaussian wave packet or vortex solutions with their center of mass at the origin. Pérez-García's group indicated that these solutions with the center of mass displaced from the origin can behave like solitons and that the inter-component interaction among solitons may yield soliton-soliton interactions. To materialize their idea, however, one must investigate a crucial role of breathing or self-similar nature of above localized structures together with suitable corrections to the phase.

To begin with, we apply the variational principle to derive from (1) the evolution equation for the collective coordinates of wavepackets. The collective coordinates are phase variables besides the center of mass and width. We Taylor-expand the phase with respect to space coordinates relative to the center of mass. The trial Gaussian wavepackets are constructed from the circularly-symmetric ground state solution of (1) with NICI suppressed:

$$\begin{aligned}\psi_j(\mathbf{r}) &= \sqrt{\frac{1}{\pi w_j^2}}\exp\left[-\frac{(\mathbf{r}-\mathbf{R}_j)^2}{2w_j^2}\right]\exp(i\Theta_j(\mathbf{r})) \\ \Theta_j(\mathbf{r}) &= \boldsymbol{\alpha}_j(\mathbf{r}-\mathbf{R}_j)+\frac{1}{2}(\mathbf{r}-\mathbf{R}_j)^T\boldsymbol{\beta}_j(\mathbf{r}-\mathbf{R}_j) \quad (2)\end{aligned}$$

(One may also choose excited states responsible for vortices.) The meaning of collective coordinates is as follows: $\mathbf{R}=(X,Y)$ is the center of mass; w is width of the circular wavepacket.

$$\boldsymbol{\alpha}=(\alpha^X,\alpha^Y) \tag{3}$$

and

$$\boldsymbol{\beta}=\begin{pmatrix}\beta^{XX} & \beta^{XY}\\ \beta^{YX} & \beta^{YY}\end{pmatrix}, \qquad \beta^{XY}=\beta^{YX} \tag{4}$$

are respectively the first- and second-order coefficients of Taylor-expansion of the phase Θ with respect to $\mathbf{r}-\mathbf{R}$. A trivial constant phase has been suppressed. The expansion of the phase in Eq.(2) is the most physical and natural, although the existing works[3, 4] employ another expansion, i.e., with respect to $\mathbf{r}$ rather than $\mathbf{r}-\mathbf{R}$.

According to the variational principle, Eq.(1) can be derived by minimizing the action function obtained from Lagrangian density for field variables,

$$\begin{aligned}\mathcal{L} &= \sum_{j=1}^{n}\left[\frac{i}{2}(\psi_j\dot{\psi}_j^*-\psi_j^*\dot{\psi}_j)+\frac{1}{2}|\nabla\psi_j|^2+V|\psi_j|^2+\frac{g}{2}|\psi_j|^4\right] \\ &\quad +\sum_{k>j}^{n}\Lambda g|\psi_j|^2|\psi_k|^2. \end{aligned}\tag{5}$$

where the asterisk denotes a complex conjugate. In fact, the multi-component GP equation is obtained from Lagrange equation:

$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\psi}_j^*}-\frac{\partial\mathcal{L}}{\partial\psi_j^*}+\nabla\frac{\partial\mathcal{L}}{\partial\nabla\psi_j^*}=0. \tag{6}$$

We now insert (2) into (5) and obtain the Lagrangian L for the collective coordinates by integrating $\mathcal{L}$ over space coordinates:

$$\begin{aligned}L &= \int\mathcal{L}d\mathbf{r} = \\ &\sum_{j=1}^{n}\Bigg[\left(\mathrm{Tr}\dot{\boldsymbol{\beta}}_j+\mathrm{Tr}(\boldsymbol{\beta}_j^T\boldsymbol{\beta}_j)\right)\frac{w_j^2}{4} \\ &- \boldsymbol{\alpha}_j\dot{\mathbf{R}}_j+\frac{(\boldsymbol{\alpha}_j)^2}{2}+\frac{\mathbf{R}_j^2}{2}+\frac{w_j^2}{2}+\left(1+\frac{g}{2\pi}\right)\frac{1}{2w_j^2}\Bigg]+ \\ &+\sum_{j>k}^{n}\frac{\Lambda g}{\pi(w_j^2+w_k^2)}e^{-\frac{(\mathbf{R}_j-\mathbf{R}_k)^2}{w_j^2+w_k^2}}.\end{aligned}\tag{7}$$

Lagrange equations of motion for the phase variables $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ lead to

$$\boldsymbol{\alpha}_j=\dot{\mathbf{R}}_j \tag{8}$$

and

$$\beta_j^X=\beta_j^Y=\frac{\dot{w}_j}{w_j}, \qquad \beta_j^{XY}=\beta_j^{YX}=0. \tag{9}$$

The issue in Eqs.(8) and (9) shows that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ adiabatically follow the center of mass $\mathbf{R}$ and width w. Therefore we can rewrite Lagrange equations for $\mathbf{R}$ and w by eliminating $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$: Equation of motion for $\mathbf{R}$,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\mathbf{R}}_j}\right)-\frac{\partial L}{\partial\mathbf{R}_j}=0 \tag{10}$$

becomes

$$\ddot{\mathbf{R}}_j+\mathbf{R}_j+\frac{\partial}{\partial\mathbf{R}_j}\left(\sum_{j>k}^{n}\frac{\Lambda g}{\pi(w_j^2+w_k^2)}e^{-\frac{(\mathbf{R}_j-\mathbf{R}_k)^2}{w_j^2+w_k^2}}\right)=0, \tag{11}$$

while the equation for w,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{w}_j}\right)-\frac{\partial L}{\partial w_j}=0, \tag{12}$$

results in

$$\begin{aligned}\ddot{w}_j &+ w_j-\frac{1}{w_j^3}\left(1+\frac{g}{2\pi}\right) \\ &+ \frac{\partial}{\partial w_j}\left(\sum_{j>k}^{n}\frac{\Lambda g}{\pi(w_j^2+w_k^2)}e^{-\frac{(\mathbf{R}_j-\mathbf{R}_k)^2}{w_j^2+w_k^2}}\right)=0.\end{aligned}\tag{13}$$

It is interesting that, although the expansion of the phase in Eq.(2) differs from the conventional ones, the resultant equations for $\mathbf{R}$ and w in case of $\Lambda g=0$ agrees with those of Pérez-García's group for a single-component BEC. However, our scheme is more logical and the results for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in (8) and (9) directly correspond to the velocity of center of mass and breathing of wavepacket, respectively.

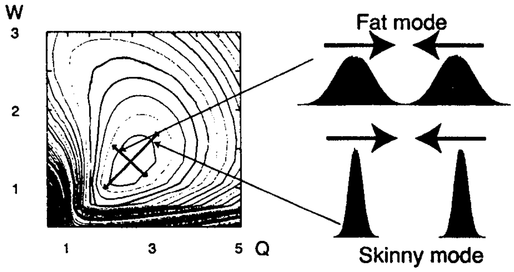

FIG. 1: Potential in w-Q space ($\Lambda = 10, g = 10\pi, M = 1$) and normal modes.

In the following, we concentrate on the two-component BEC with a harmonic trap in two dimensions and explore the stability of wavepacket breathing against NICI (Λg). Effective Lagrangian leading to Eqs.(11) and (13) is expressed as

$$L_{\text{eff}} = L_w + \frac{1}{2}\sum_{j=1}^{2}(\dot{\mathbf{R}}_j^2 - R_j^2) - \frac{\Lambda g}{\pi(w_1^2 + w_2^2)}e^{-\frac{(\mathbf{R}_1-\mathbf{R}_2)^2}{w_1^2+w_2^2}} \tag{14}$$

with

$$L_w = \frac{1}{2}\sum_{j=1}^{2}\left[\dot{w}_j^2 - w_j^2 - \frac{1}{2w_j^2}\left(1+\frac{g}{2\pi}\right)\right]. \tag{15}$$

Let us define the center of mass of two components $\mathbf{R}_0$ and the relative displacement $\mathbf{Q}$ as:

$$\mathbf{R}_0 = \frac{\mathbf{R}_1 + \mathbf{R}_2}{\sqrt{2}}, \qquad \mathbf{Q} = \frac{\mathbf{R}_2 - \mathbf{R}_1}{\sqrt{2}} \tag{16}$$

and suppress the global translational degree of freedom ($\mathbf{R}_0 = \dot{\mathbf{R}}_0 = 0$). Further, because of rotational symmetry, the angular momentum is a constant of motion:

$$M = \frac{\partial L}{\partial \dot{\theta}} = Q^2\dot{\theta}. \tag{17}$$

θ denotes a polar angle for $\mathbf{Q}$. Choosing a synchronous width dynamics ($w_1 = w_2 = w$) with the canonical momentum $p_w = \frac{\partial L}{\partial \dot{w}}$, the effective Hamiltonian is given by

$$H_{\text{eff}} = \frac{1}{2}P_Q^2 + \frac{1}{4}p_w^2 + V(Q, w)$$

with

$$V(Q,w) = \frac{1}{2}Q^2 + \frac{M^2}{2Q^2} + w^2 + \frac{1}{w^2}\left(1+\frac{g}{2\pi}\right) + \frac{\Lambda g}{2\pi w^2}e^{-\frac{Q^2}{w^2}}. \tag{18}$$

We see that the competition between a centrifugal force due to the non-zero M and the harmonic trap leads to an equilibrium distance Q_0. In the absence of NICI ($\Lambda = 0$) both w and Q show independent elliptic motions around the stable fixed points (minima of the individual potentials, $w = w_0 = (1+g/2\pi)^{1/4}$ and $Q = Q_0 = \sqrt{M}$) which has the common frequency($\omega_Q = 2, \omega_w = \omega_w^0 = 2$). This means that decrease in the relative displacement has no effect on the breathing oscillation of the width, which provides another logical basis for supporting Ehrenfest theorem for the single-component BEC.

On the contrary, in the presence of nonvanishing NICI, the width and relative displacement will constitute compound normal modes: In "fat or optical mode", the decrease of Q is accompanied by the increase of w, while in "skinny or acoustic mode", Q and w change synchronously (see Fig.1). The emergence of the fat mode leads to the wave interference between two wavepackets whenever Q decreases, invalidating the effective nonlinear dynamics based on the particle picture.

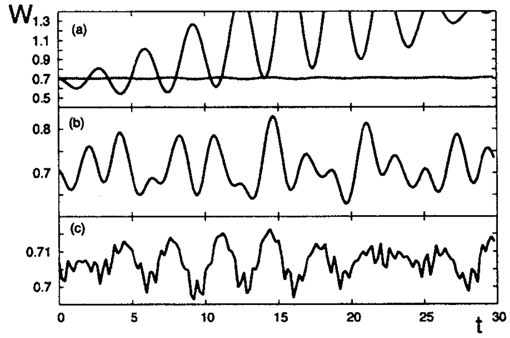

FIG. 2: Time evolution of wavepacket width: Ω dependence. $\Lambda g = 3$. (a) $\Omega = 0$(solid line) and $\Omega = 100$(dotted line), (b)$\Omega = 3$,(c)$\Omega = 10$.

We here introduce a time-periodic NICI defined by

$$g_{12} = \Lambda g \cos(\Omega t) \tag{19}$$

rather than the static one ($g_{12} = \Lambda g$). If $\Omega >> \omega_w^0$ with $\omega_w^0 (= 2)$ for a typical frequency of breathing of wavepacket, the system has a vanishing NICI in the time-averaging sense, and each of wavepackets is expected to show the stable and regular breathing oscillation. When Ω is decreased but keeps a value larger than ω_w^0, the breathing oscillation will be able to show no signature of blow-up. However, when $\Omega < \omega_w^0$, the breathing oscillation becomes unstable and wave interference occurs, breaking an interacting particle picture for the multi-component BEC. The above conjecture is supported in our direct numerical interaction of the two-component GP equation. Fig.2(a) shows a blow-up of the breathing oscillation for $\Omega = 0$ and a recovery of stable oscillation for $\Omega = 100$, and Figs.2(b) and 2(c) show stable oscillations for $\Omega = 3$ and 10.

Under a fixed value of Ω in the range $\Omega > \omega_w^0$, we find the transition from regular to chaotic oscillations of the breathing when the amplitude Λg is increased. In fact, in the case of $\Omega = 5$, the oscillation is irregular for $\Lambda g = 2$ and 4 (see Figs.3(b) and 3(c)), while it is

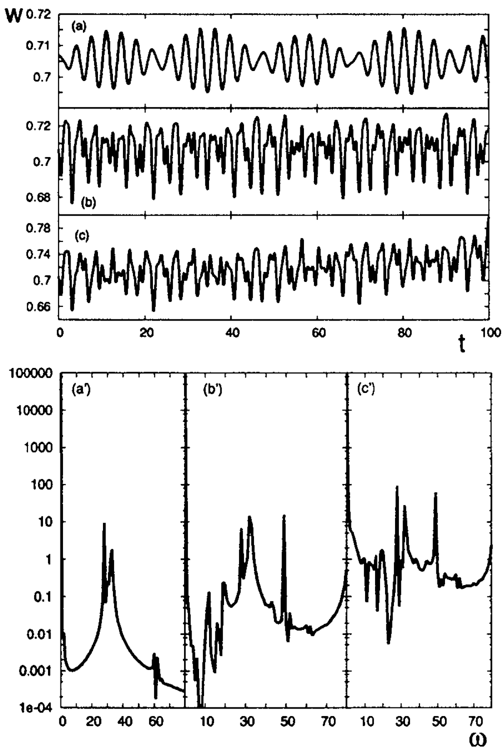

FIG. 3: Time evolution of wavepacket width: Λg dependence(upper panel) and Power spectrum(lower panel). $\Omega = 5$. (a)(a')$\Lambda g = 0$,(b)(b')$\Lambda g = 2$, (c)(c')$\Lambda g = 4$.

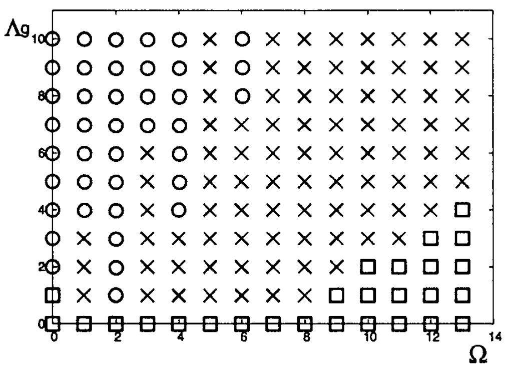

FIG. 4: Phase diagram in Λg-Ω space. $\bigcirc, \triangle, \square$ stands for blow-up, chaotic and regular regions, respectively.

regular for $\Lambda g = 0$ (see Fig.3(a)). The corresponding broad power spectrum characterizing chaotic oscillation is given in Fig.3(b') and 3(c'), while the line spectrum corresponding to Fig.3(a) is given in Fig.3(a').

We have systematically investigated the wave dynamics in the two-component BEC under the oscillating NCIC. Figure4 is a phase diagram in Λg-Ω space, which shows three distinct regions, (i)blow-up, (ii)stable and chaotic and (iii) stable and regular regions. The result suggests that two-component BEC with harmonic trap under the oscillating NICI is clearly characterized by an interacting particle picture which describes the transition from regular to chaotic motions.

To develop the effective nonlinear dynamics in the multi-component BEC with the harmonic trap in two dimensions, we have examined a novel idea proposed by Pérez-García's group. Firstly we assign to a particle the Gaussian wavepacket for each BEC component and have obtained the effective equation of motion for the width and inter-particle separation by using the refined collective coordinate method. The inter-component interaction (ICI) is found to generate normal modes that conbine the breathing oscillation of each wavepacket with relative displacement between wavepackets. In "fat or optical mode", decrease of the displacement yields increase of the width of wavepacket. This fact leads to breaking of a particle picture of Pérez-García's group for the multi-component BEC. We propose and show that a high-frequency time-periodic ICI with zero average stabilizes the breathing motion of each wavepacket. Provided that the frequency is larger than the characteristic breathing frequency, one can also see the interesting transition from regular to chaotic oscillations as the amplitude of ICI is increased. We have shown this transition in 3-body problem, i.e., in the case of 3 component BEC. In the case of the attractive BEC, the oscillating nonlinearity is known to stabilize high-dimensional wavepackets in the absence of harmonic trap. In the similar way, in the case of the multi-component BEC with harmonic trap, the analogous picture holds for the high-dimensional wavepackets which are mutually interacting through time-dependent ICI with zero average and non-zero variance. The variance is found to control the chaoticity of breathing motions of wavepackets.

This talk is based on the latest joint work with H.Yamasaki and Y.Natsume.

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